Auctioning Keywords in Online Search

Jianqing Chen
The University of Calgary
jiachen@ucalgary.ca

De Liu
University of Kentucky
de.liu@uky.edu

Andrew B. Whinston
University of Texas at Austin
abw@uts.cc.utexas.edu

Web Appendix

Derivation of the Expected Revenue. Denote $V(v)$ as the equilibrium payoff of an advertiser of type v, and $P_j(v)$ as the equilibrium probability for him to win the jth share. So

(W1) $$V(v) = v\sum_{j=1}^{n} Q(s_j)P_j(v) - E\left[\text{equilibrium payment of type } v\right]$$

First notice that

(W2) $$V(v) \geq V(\tilde{v}) + (v - \tilde{v})\sum_{j=1}^{n} Q(s_j)P_j(\tilde{v}),$$

where the right-hand side is the payoff of a type-v bidder if he follows instead an equilibrium strategy of a type-$\tilde{v}$ bidder.

Therefore we have

(W3) $$V(v) \geq V(v + dv) + (-dv)\sum_{j=1}^{n} Q(s_j)P_j(v + dv)$$

(W4) $$V(v + dv) \geq V(v) + dv\sum_{j=1}^{n} Q(s_j)P_j(v)$$

Reorganizing the above two equations yields

(W5) $$\sum_{j=1}^{n} Q(s_j)P_j(v + dv) \geq \frac{V(v + dv) - V(v)}{dv} \geq \sum_{j=1}^{n} Q(s_j)P_j(v)$$

Taking the limit of dv to zero, we obtain $\frac{dV(v)}{dv} = \sum_{j=1}^{n} Q(s_j)P_j(v)$. Moving dv to the right hand side, integrating both sides from $\underline{v}$ to v, and assuming $V(\underline{v}) = 0$ (the lowest type gets zero payoff), we get

$$V(v) = \sum_{j=1}^{n} Q(s_j) \int_{\underline{v}}^{v} P_j(x)dx, \text{ for } v \in [\underline{v}, \bar{v}]. \tag{W6}$$

Notice the expected payment from an advertiser of type v is $v \sum_{j=1}^{n} Q(s_j)P_j(v) - V(v)$ by (W1). So the expected payment from one bidder is

$$\begin{aligned} E\left[v\sum_{j=1}^{n} Q(s_j)P_j(v) - V(v)\right] &= \int_{\underline{v}}^{\bar{v}} \left[v\sum_{j=1}^{n} Q(s_j)P_j(v) - \sum_{j=1}^{n} Q(s_j)\int_{\underline{v}}^{v} P_j(t)dt\right] f(v)dv \\ &= \int_{\underline{v}}^{\bar{v}} \left[v\sum_{j=1}^{n} Q(s_j)P_j(v)f(v) - (1-F(v))\sum_{j=1}^{n} Q(s_j)P_j(v)\right] dv \\ &= \sum_{j=1}^{n} Q(s_j)\int_{\underline{v}}^{\bar{v}} P_j(v)\left(v - \frac{1-F(v)}{f(v)}\right) f(v)dv \qquad \text{(W7)} \end{aligned}$$

The total expected revenue from all advertisers is n times the above. With strictly increasing bidding functions, the equilibrium probability for an advertiser of type v to win the jth share is the likelihood that $j-1$ of his competitors have higher valuation than his and the rest of the competitors have lower valuation, and thus $P_j(v)$ can be specified as in (7). ■

Proof of Lemma 1. (a) For $j = 1, 2, ..., n-1$,

$$\begin{aligned} \alpha_j &= n\int_{\underline{v}}^{\bar{v}} P_j(v)\left[vf(v) - (1-F(v))\right] dv = n\int_{\underline{v}}^{\bar{v}} P_j(v)\, d\left[-v(1-F(v))\right] \\ &= n\int_{\underline{v}}^{\bar{v}} v(1-F(v))\, dP_j(v) \\ &= n\int_{\underline{v}}^{\bar{v}} \tbinom{n-1}{n-j} F(v)^{n-j-1}(1-F(v))^{j-1}\left[(n-j) - (n-1)F(v)\right] vf(v)\, dv \qquad \text{(W8)} \end{aligned}$$

where the third step is due to integration by parts. We can easily verify $\alpha_1 > 0$.

For $j = 2, 3, ..., n-1,$

$$\alpha_1 - \alpha_j = n \int_{\underline{v}}^{\bar{v}} \{(n-1) F(v)^{n-2} (1 - F(v)) - \binom{n-1}{n-j} F(v)^{n-j-1} (1 - F(v))^{j-1} [(n-j) - (n-1) F(v)]\} v f(v)\, dv \tag{W9}$$

Denoting $A(v) \equiv (n-1) F(v)^{j-1} - \binom{n-1}{n-j} (1 - F(v))^{j-2} [(n-j) - (n-1) F(v)]$, we can rewrite (W9) as $\alpha_1 - \alpha_j = n \int_{\underline{v}}^{\bar{v}} [1 - F(v)] F(v)^{n-j-1} A(v) v f(v)\, dv$. We argue that $A(v)$ single-crosses zero from below on $[\underline{v}, \bar{v}]$. To see, let v^0 be the solution to $(n-j) - (n-1) F(v) = 0$. We can verify that $A(\underline{v}) < 0$, $A(v)$ increases in v for $v \leq v^0$, and $A(v)$ is positive for all $v > v^0$. Thus $A(v)$ crosses zero only once from below, implying $[1 - F(v)] F(v)^{n-j-1} A(v) f(v)$ also single-crosses zero from below on $(\underline{v}, \bar{v})$. Denoting the crossing point of the latter as v^c, we have

$$\begin{aligned} \alpha_1 - \alpha_j &= n \int_{\underline{v}}^{\bar{v}} (1 - F(v)) F(v)^{n-j-1} A(v) f(v) v dv \\ &> n v^c \int_{\underline{v}}^{\bar{v}} (1 - F(v)) F(v)^{n-j-1} A(v) f(v)\, dv \\ &= n v^c \int_{\underline{v}}^{\bar{v}} [P_2(v) - j P_{j+1}(v) + (j-1) P_j(v)] f(v)\, dv \end{aligned} \tag{W10}$$

where the last equality results from substituting the definition of $A(v)$ and rearranging terms. The right side of (W10) is zero because for $j = 1, ..., n,$

$$\begin{aligned} \int_{\underline{v}}^{\bar{v}} P_j(v) f(v)\, dv &= \binom{n-1}{n-j} \int_{\underline{v}}^{\bar{v}} F(v)^{n-j} [1 - F(v)]^{j-1} dF(v) \\ &= \binom{n-1}{n-j} \int_0^1 x^{n-j} (1-x)^{j-1} dx \\ &= \binom{n-1}{n-j} \binom{n-1}{n-j}^{-1} \frac{1}{n} = \frac{1}{n} \end{aligned} \tag{W11}$$

where the second step is due to integration by substitution and the third step is due to repeated integration by parts. Therefore, $\alpha_1 - \alpha_j > 0$ for $j = 2, 3, ..., n-1$.

We next show that $\alpha_1 - \alpha_n > 0$.

$$\begin{aligned}
\alpha_1 - \alpha_n &= n\int_{\underline{v}}^{\bar{v}} \left\{F(v)^{n-1} - [1-F(v)]^{n-1}\right\} d\left[-v(1-F(v))\right] \\
&= -n\underline{v} + n\int_{\underline{v}}^{\bar{v}} v\left[1-F(v)\right](n-1)\left[F(v)^{n-2} + (1-F(v))^{n-2}\right] f(v)dv \\
&> -n\underline{v} + n\underline{v}\int_{\underline{v}}^{\bar{v}} \left[1-F(v)\right](n-1)\left[F(v)^{n-2} + (1-F(v))^{n-2}\right] f(v)dv \\
&= -n\underline{v} + n\underline{v}\left(\frac{1}{n} + \frac{n-1}{n}\right) = 0
\end{aligned}$$

where the second step is due to integration by parts and the fourth step is due to (W11).

(b) Denote $h_j(x) \equiv nP_j(x) f(x)$. By (W11), $\int_{\underline{v}}^{\bar{v}} h_j(x)\,dx = 1$. Thus we can regard $h_j(x)$ as a probability density function. We next show that for $j = 1, 2, ..., n-1$, $h_j(x)$ first-order stochastically dominates $h_{j+1}(x)$.

$$\begin{aligned}
&h_j(x) - h_{j+1}(x) \\
&= nf(x)\left\{\tbinom{n-1}{n-j}F(x)^{n-j}\left[1-F(x)\right]^{j-1} - \tbinom{n-1}{n-j-1}F(x)^{n-j-1}\left[1-F(x)\right]^{j}\right\} \\
&= \tbinom{n}{j}f(x)\,F(x)^{n-j-1}\left[1-F(x)\right]^{j-1}\left[nF(x) - (n-j)\right]
\end{aligned}$$

Denote v_j^c as the solution to $nF(x) - (n-j) = 0$. Because $h_j(x) < h_{j+1}(x)$ for any $x \in \left(\underline{v}, v_j^c\right)$, $\int_{\underline{v}}^{v} h_j(x)\,dx < \int_{\underline{v}}^{v} h_{j+1}(x)\,dx$ for $v \in \left(\underline{v}, v_j^c\right)$. Because $h_j(x) > h_{j+1}(x)$ for any $x \in \left(v_j^c, \bar{v}\right)$, $\int_{v}^{\bar{v}} h_j(x)\,dx > \int_{v}^{\bar{v}} h_{j+1}(x)\,dx$ for $v \in \left(v_j^c, \bar{v}\right)$, which implies $\int_{\underline{v}}^{v} h_j(x)\,dx < \int_{\underline{v}}^{v} h_{j+1}(x)\,dx$ for $v \in \left(v_j^c, \bar{v}\right)$ (note that $\int_{\underline{v}}^{v} h_j(x)\,dx = 1 - \int_{v}^{\bar{v}} h_j(x)\,dx$). In all, we have $\int_{\underline{v}}^{v} h_j(x)\,dx < \int_{\underline{v}}^{v} h_{j+1}(x)\,dx$ for any $v \in (\underline{v}, \bar{v})$, implying that $h_j(x)$ first-order stochastically dominates $h_{j+1}(x)$. According to the property of first-order stochastic dominance (e.g., Proposition 6.D.1 at page 195 of Mas-Colell, Whinston, and Green (1995)), if $J(x)$ is an increasing function of x, $\int_{\underline{v}}^{\bar{v}} h_j(x)\,J(x)\,dx > \int_{\underline{v}}^{\bar{v}} h_{j+1}(x)\,J(x)\,dx$. Therefore $\alpha_j > \alpha_{j+1}$.

■

Proof of Lemma 2. Assume the optimal share structure is $(s_1^*, s_2^*, ..., s_n^*)$. Denote $\sum_{j=j_k+1}^{j_{k+1}} s_j^* \equiv \sigma$ and notice that $s_{j_k}^* \geq \frac{1}{j_{k+1}-j_k}\sigma \geq s_{j_{k+1}+1}^* \geq 0$ because of the size-order constraint. $(s_{j_k+1}^*, s_{j_k+2}^*, ..., s_{j_{k+1}}^*)$ must be the solution to the following maximization problem:

$$\text{(W12)} \qquad \max \sum_{j=j_k+1}^{j_{k+1}} \alpha_j Q(s_j), \text{ subject to:} \qquad s_{j_k+1} \geq ... \geq s_{j_{k+1}} \text{ and } \sum_{j=j_k+1}^{j_{k+1}} s_j \leq \sigma$$

$$\text{(W13)} \qquad s_{j_k}^* \geq s_{j_k+1} \text{ and } s_{j_{k+1}} \geq s_{j_{k+1}+1}^*$$

We will work on the related maximization problem without constraint (W13) and check (W13) later. The Lagrangian function then can be written as

$$L = \sum_{j=j_k+1}^{j_{k+1}} \alpha_j Q(s_j) + \mu \left(\sigma - \sum_{j=j_k+1}^{j_{k+1}} s_j \right) + \sum_{j=j_k+1}^{j_{k+1}-1} \gamma_j (s_j - s_{j+1})$$

where μ and γ_j are Lagrange multipliers. Hence, the Kuhn-Tucher conditions are (let $\gamma_{j_k} \equiv 0$, $\gamma_{j_{k+1}} \equiv 0$)

$$\text{(W14)} \qquad \alpha_j Q'(s_j) - \mu + \gamma_j - \gamma_{j-1} = 0, \text{ for } j = j_k + 1, ..., j_{k+1}$$

Averaging (W14) for the first l shares and the remaining shares, respectively, we have

$$\text{(W15)} \qquad \frac{1}{l}\left(\sum_{j=j_k+1}^{j_k+l} \alpha_j Q'(s_j) + \gamma_{j_k+l}\right) = \frac{1}{j_{k+1} - j_k - l}\left(\sum_{j=j_k+l+1}^{j_{k+1}} \alpha_j Q'(s_j) - \gamma_{j_k+l}\right).$$

By definition, j_{k+1} is the maximizer for the average return factor starting from $j_k + 1$, so

$$\text{(W16)} \qquad \frac{1}{l}\sum_{j=j_k+1}^{j_k+l} \alpha_j \leq \frac{1}{j_{k+1} - j_k - l}\sum_{j=j_k+l+1}^{j_{k+1}} \alpha_j$$

Also note that $Q'(s_j)$ is nondecreasing in j. Therefore, we have $\frac{1}{l}\sum_{j=j_k+1}^{j_k+l} \alpha_j Q'(s_j) \leq \frac{1}{j_{k+1}-j_k-l}\sum_{j=j_k+l+1}^{j_{k+1}} \alpha_j Q'(s_j)$. If $\gamma_{j_k+l} = 0$, (W15) can hold only if (W16) holds in equality and $s_{j_k+1} = ... = s_{j_{k+1}}$. In other words, if any $\gamma_j = 0$ ($j_k < j < j_{k+1}$), we must have $s_{j_k+1} = ... = s_{j_{k+1}}$. Otherwise, we have $\gamma_j > 0$ for all $j_k < j < j_{k+1}$, which implies $s_{j_k+1} = ... = s_{j_{k+1}}$

by the Kuhn-Tucker condition. So, regardless, we have $s_{j_k+1} = ... = s_{j_{k+1}} = \frac{1}{j_{k+1}-j_k}\sigma$, which naturally satisfies constraint (W13). ■

Proof of Lemma 3. It is sufficient to show that if $\hat{s}_j \geq s_j$ then $\hat{s}_{j+1} \geq s_{j+1}$, for any j. If j and $j+1$ are located in the same plateau, $\hat{s}_j = \hat{s}_{j+1}$ and $s_j = s_{j+1}$, and $\hat{s}_{j+1} \geq s_{j+1}$ holds trivially. So we assume j and $j+1$ are located in plateau k and $k+1$, respectively. If $s_{j+1} = 0$, the result holds trivially. Suppose $s_{j+1} > 0$ (so that $\hat{s}_j \geq s_j > s_{j+1} > 0$). By Proposition 2, we have $\hat{\bar{\alpha}}_k Q'(\hat{s}_j) \geq \hat{\bar{\alpha}}_{k+1} Q'(\hat{s}_{j+1})$ and $\bar{\alpha}_k Q'(s_j) = \bar{\alpha}_{k+1} Q'(s_{j+1})$. Together with condition (17), we have

$$\text{(W17)} \qquad \frac{Q'(\hat{s}_j)}{Q'(\hat{s}_{j+1})} \geq \frac{\hat{\bar{\alpha}}_{k+1}}{\hat{\bar{\alpha}}_k} \geq \frac{\bar{\alpha}_{k+1}}{\bar{\alpha}_k} = \frac{Q'(s_j)}{Q'(s_{j+1})}$$

which implies $\frac{Q'(\hat{s}_j)}{Q'(s_j)} \geq \frac{Q'(\hat{s}_{j+1})}{Q'(s_{j+1})}$. Note that $\frac{Q'(\hat{s}_j)}{Q'(s_j)} \leq 1$ by concavity of $Q(\cdot)$ and $\hat{s}_j \geq s_j$. So we have $\frac{Q'(\hat{s}_{j+1})}{Q'(s_{j+1})} \leq 1$, which implies $\hat{s}_{j+1} \geq s_{j+1}$. ■

Lemma W1 (Ranking of $\alpha_j(v_0)$) *Under the MHR condition, for any marginal type $v_0 \in (\underline{v}, \bar{v})$, if $\alpha_j(v_0) > 0$, $\alpha_j(v_0) > \alpha_{j+1}(v_0)$; if $\alpha_j(v_0) \leq 0$, $\alpha_{j+1}(v_0) < 0$.*

Proof. For the case $v_0 \in (\underline{v}, \bar{v})$, define $h_j(x|x \geq v_0) \equiv \frac{h_j(x)}{\int_{v_0}^{\bar{v}} h_j(t)dt}$. Following steps in the proof of Lemma 1(b), we can similarly show that $h_i(x|x \geq v_0)$ first-order stochastically dominates $h_{i+1}(x|x \geq v_0)$. Thus,

$$\text{(W18)} \qquad \int_{v_0}^{\bar{v}} h_j(x|x \geq v_0) J(x)\, dx > \int_{v_0}^{\bar{v}} h_{j+1}(x|x \geq v_0) J(x)\, dx, \text{ for } j = 1, 2, ..., n-1.$$

Substituting $h_j(x|x \geq v_0)$ with $\frac{h_j(x)}{\int_{v_0}^{\bar{v}} h_j(t)dt}$ and rearranging, we have

(W19)

$$\alpha_j(v_0) = \int_{v_0}^{\bar{v}} h_j(x) J(x)\, dx > \frac{\int_{v_0}^{\bar{v}} h_j(t)\, dt}{\int_{v_0}^{\bar{v}} h_{j+1}(t)\, dt} \int_{v_0}^{\bar{v}} h_{j+1}(x) J(x)\, dx = \frac{\int_{v_0}^{\bar{v}} h_j(t)\, dt}{\int_{v_0}^{\bar{v}} h_{j+1}(t)\, dt} \alpha_{j+1}(v_0).$$

Suppose $\alpha_j(v_0) > 0$. If $\alpha_{j+1}(v_0) \geq 0$, from $\int_{v_0}^{\bar{v}} h_j(t)\, dt > \int_{v_0}^{\bar{v}} h_{j+1}(t)\, dt > 0$ (because $h_j(t)$ first-order stochastically dominates $h_{j+1}(t)$), we have $\alpha_j(v_0) > \alpha_{j+1}(v_0)$. If $\alpha_{j+1}(v_0) < 0$, it is easy to see $\alpha_j(v_0) > \alpha_{j+1}(v_0)$.

If $\alpha_j(v_0) \leq 0$, (W19) implies $\alpha_{j+1}(v_0) < 0$. ■

References

[1] Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green (1995), *Microeconomic Theory*. Oxford Unversity Press, New York.