Web Appendix for "Product Variety, Informative Advertising and Price Competition"

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APPENDIX A: TECHNICAL NOTES

Derivation of Equilibrium Price

We focus on the symmetric pure strategy Nash equilibrium to understand optimal firm behavior. We first derive the optimal price p_j^* and show that it is a local optimum (Claim 1). Then we partition consumer valuation v into four regions and establish that the firm cannot profitably deviate from p_j^* in any of the four regions (Claims 2–5). Finally, based on these claims, we present the equilibrium price in different regions of consumer valuation.

Firm j's profit maximization problem is:

$$\max_{p_j} p_j q_j - A(\phi;\alpha) \tag{A–1}$$

where q_j consists of the demand emanating from the four consumer groups discussed in Section 2. The demand from each of the four groups is as follows:

Group 1:

$$\frac{2}{N}\frac{1}{n-1}\phi\left[1-(1-\phi)^{n-1}\right]\sum_{k\neq j,k\in\{1,\ldots,n\}}\max\left\{\min\left\{\frac{1}{2}+\frac{p_k-p_j}{2t},1\right\},0\right\}. \tag{A–2}$$

Group 2:

$$\frac{2}{N}\phi(1-\phi)^{n-1}\min\left\{\max\left\{0,\frac{v-p_j}{t}\right\},\frac{1}{2}\right\}. \tag{A–3}$$

Group 3:

$$\frac{2}{N}\frac{N-n}{n}\left[1-(1-\phi)^{n}\right]\min\left\{\max\left\{0,\frac{v-p_j}{t}-\frac{1}{2}\right\},\frac{1}{2}\right\}. \tag{A–4}$$

Group 4:

$$\frac{2}{N}\frac{1}{n-1}(1-\phi)\left[1-(1-\phi)^{n-1}\right]\sum_{k\neq j,k\in\{1,\ldots,n\}}\min\left\{\max\left\{0,\frac{v-p_j}{t}-\frac{1}{2}\right\},\frac{1}{2}\right\}. \tag{A–5}$$

As firms are symmetric, the demand from Group 1 (equation A–2) and Group 2 (equation A–3) can be simplified to $\frac{1}{N}\phi\left[1-(1-\phi)^{n-1}\right]\left(1+\frac{p_k-p_j}{t}\right)$ and $\frac{1}{N}\phi(1-\phi)^{n-1}$, respectively, when $\frac{1}{2}<\frac{v-p_j}{t}<1$. After simplifying the demand from Groups 3 and 4 along similar lines and summing the demand from the four groups of consumers, we obtain q_j, which is given by:

$$q_j=\begin{cases}\frac{1}{N}\phi\left[1-(1-\phi)^{n-1}\right]\left(1+\frac{p_k-p_j}{t}\right)+\frac{1}{N}\phi(1-\phi)^{n-1} \\ \quad+\frac{2}{N}\left\{\begin{array}{c}\frac{N-n}{n}\left[1-(1-\phi)^{n}\right]\\ +(1-\phi)\left[1-(1-\phi)^{n-1}\right]\end{array}\right\}\left(\frac{v-p_j}{t}-\frac{1}{2}\right) & \text{for } \frac{1}{2}<\frac{v-p_j}{t}<1 \\ \frac{1}{N}\phi\left[1-(1-\phi)^{n-1}\right]\left(1+\frac{p_k-p_j}{t}\right)+\frac{1}{N}\phi(1-\phi)^{n-1} \\ \quad+\frac{1}{N}\left\{\frac{N-n}{n}\left[1-(1-\phi)^{n}\right]+(1-\phi)\left[1-(1-\phi)^{n-1}\right]\right\} & \text{for } \frac{v-p_j}{t}\geq 1\end{cases} \tag{A–6}$$

We initially assume that advertising reach ϕ is exogenous to the model (and make it endogenous when we consider advertising technology). Hence, as $A(\phi;\alpha)$ is also exogenous, we differentiate revenues with respect to p_j, set it equal to zero ($\frac{\partial R_j}{\partial p_j}=0$) and solve for p_j^*. We have:

$$p_j^*=\begin{cases}\frac{N(2v-t)[1-(1-\phi)^n]-2n\phi(v-t)}{4N[1-(1-\phi)^n]-n\phi(1-\phi)^{n-1}-3n\phi} & \text{for } \frac{1}{2}<\frac{v-p_j}{t}<1 \\ \frac{Nt[1-(1-\phi)^n]}{n\phi\left[1-(1-\phi)^{n-1}\right]} & \text{for } \frac{v-p_j}{t}\geq 1\end{cases} \tag{A–7}$$

On plugging back p_j^* into R_j and simplifying, we obtain:

$$R_j^*=\begin{cases}\frac{\{N(2v-t)[1-(1-\phi)^n]-2n\phi(v-t)\}^2\left\{2N[1-(1-\phi)^n]-n\phi\left[1+(1-\phi)^{n-1}\right]\right\}}{Nnt\left\{4N[1-(1-\phi)^n]-n\phi\left[3+(1-\phi)^{n-1}\right]\right\}^2} & \text{for } \frac{1}{2}<\frac{v-p_j}{t}<1 \\ \frac{Nt[1-(1-\phi)^n]^2}{n^2\phi\left[1-(1-\phi)^{n-1}\right]} & \text{for } \frac{v-p_j}{t}\geq 1\end{cases} \tag{A–8}$$

Next we prove five claims that help to establish the equilibrium prices for different regions in consumer valuation.

Claim 1 *The prices given in equation (A–7) are local maxima.*

Proof. When $\frac{1}{2}<\frac{v-p_j}{t}<1$, the second order condition for an interior solution is satisfied, as

$$\begin{aligned}\frac{\partial^2 R_j}{\partial p_j^2} &= \frac{-2}{Nnt}\left\{2N\left[1-(1-\phi)^n\right]-n\phi\left[1+(1-\phi)^{n-1}\right]\right\} \\ &\leq 0.\end{aligned} \tag{A–9}$$

Similarly, when $\frac{v-p_j}{t} \geq 1$, we have:

$$\frac{\partial^2 R_j}{\partial p_j^2} = \frac{-2\phi}{Nt}\left[1-(1-\phi)^{n-1}\right] \leq 0 \tag{A–10}$$

Hence, the prices reported in equation (A–7) are optimal. ■

Next, we proceed to establish that it is not possible for the firm to profitably deviate from the optimal prices p_j^* in the four different regions of v.

Claim 2 *The exists a unique interior solution to p_j^* in Region 4 ($v_3 < v \leq v_4$).*

Proof. When $\frac{v-p_j}{t} \geq 1$, we need to verify that the market is fully covered. Otherwise, firm j is better off charging the price $v-t$ and selling to those who have one alternative only. Note that consumers in groups 2, 3, and 4 (equations A–3, A–4, and A–5) have only one one alternative to buy. Now suppose that *only* firm j defects and charges $p_j = v-t$. Firm j's revenue from these consumers is given by:

$$\begin{aligned}\widehat{R} &= (v-t)\left\{\frac{1}{N}\phi(1-\phi)^{n-1} + \frac{1}{N}\left\{\begin{array}{c}\frac{N-n}{n}\left[1-(1-\phi)^n\right] \\ +(1-\phi)\left[1-(1-\phi)^{n-1}\right]\end{array}\right\}\right\} \\ &= \frac{v-t}{Nn}\left\{N\left[1-(1-\phi)^n\right] - n\phi\left[1-(1-\phi)^{n-1}\right]\right\}\end{aligned} \tag{A–11}$$

For the market to be fully covered, $\widehat{R}$ should be less than the equilibrium revenue. In other words, the deviation from equilibrium price should not be profitable. That is,

$$\widehat{R} \leq \frac{Nt\left[1-(1-\phi)^n\right]^2}{n^2\phi\left[1-(1-\phi)^{n-1}\right]}. \tag{A–12}$$

This condition will be satisfied if:

$$v \leq v_4 \equiv t + \frac{N^2 t\left[1-(1-\phi)^n\right]^2}{n\phi\left\{N\left[1-(1-\phi)^n\right] - n\phi\left[1-(1-\phi)^{n-1}\right]\right\}\left[1-(1-\phi)^{n-1}\right]} \tag{A–13}$$

As $\frac{v-p_j}{t} \geq 1$, it implies that $v \geq p+t$. Hence the lower bound is given by:

$$v \geq v_3 \equiv t + \frac{Nt\left[1-(1-\phi)^n\right]}{n\phi\left[1-(1-\phi)^{n-1}\right]} \tag{A–14}$$

So the market will be fully covered if $v_3 < v \leq v_4$. In other words, $p^* = \frac{Nt[1-(1-\phi)^n]}{n\phi\left[1-(1-\phi)^{n-1}\right]}$ is the equilibrium price when $v_3 < v \leq v_4$. ■

Claim 3 *There exists a unique corner solution to p_j^* in Region 3 ($v_2 < v \leq v_3$).*

Proof. Note that for $p^* = v - t$ to be an equilibrium, we need to show that any deviation is unprofitable. That is, for p_j slightly higher than $v - t$ we should have:

$$q_j + p_j \frac{\partial q_j}{\partial p_j} = q_j + (v - t) \frac{\partial q_j}{\partial p_j} \leq 0 \tag{A–15}$$

Correspondingly, for p_j slightly lower than $v - t$ we should have:

$$q_j + p_j \frac{\partial q_j}{\partial p_j} = q_j + (v - t) \frac{\partial q_j}{\partial p_j} \geq 0 \tag{A–16}$$

$$q_j = \begin{cases} \begin{array}{c} \frac{1}{N}\phi \left[1 - (1-\phi)^{n-1}\right] \left(1 + \frac{p_k - p_j}{t}\right) + \frac{1}{N}\phi (1-\phi)^{n-1} \\ + \frac{2}{N} \left\{ \begin{array}{c} \frac{N-n}{n} \left[1 - (1-\phi)^n\right] \\ + (1-\phi)\left[1 - (1-\phi)^{n-1}\right] \end{array} \right\} \left(\frac{v - p_j}{t} - \frac{1}{2}\right) \end{array} & \text{for} & \begin{array}{c} p_j \text{ slightly} \\ \text{above } v - t \end{array} \\ \begin{array}{c} \frac{1}{N}\phi \left[1 - (1-\phi)^{n-1}\right] \left(1 + \frac{p_k - p_j}{t}\right) + \frac{1}{N}\phi (1-\phi)^{n-1} \\ + \frac{1}{N} \left\{ \frac{N-n}{n} \left[1 - (1-\phi)^n\right] + (1-\phi)\left[1 - (1-\phi)^{n-1}\right] \right\} \end{array} & \text{for} & \begin{array}{c} p_j \text{ slightly} \\ \text{below } v - t \end{array} \end{cases} \tag{A–17}$$

When p_j is slightly higher than $v - t$,

$$q_j + p_j \frac{\partial q_j}{\partial p_j} = \frac{1}{n}\left[1 - (1-\phi)^n\right] - (v - t)\frac{1}{N} \left\{ \begin{array}{c} (2-\phi)\left[1 - (1-\phi)^{n-1}\right] \\ +2\frac{N-n}{n}\left[1 - (1-\phi)^n\right] \end{array} \right\} \tag{A–18}$$

It then follows that for $v \geq v_2 \equiv t + \frac{Nt[1-(1-\phi)^n]}{n\left\{(2-\phi)\left[1-(1-\phi)^{n-1}\right] + \frac{2}{n}(N-n)[1-(1-\phi)^n]\right\}}$,

$$q_j + p_j \frac{\partial q_j}{\partial p_j} \leq 0. \tag{A–19}$$

When p_j is slightly lower than $v - t$,

$$q_j + p_j \frac{\partial q_j}{\partial p_j} = \frac{1}{n}\left[1 - (1-\phi)^n\right] - (v - t)\frac{1}{N}\phi\left[1 - (1-\phi)^{n-1}\right] \tag{A–20}$$

Hence for $v \leq v_3 \equiv t + \frac{Nt[1-(1-\phi)^n]}{n\phi\left[1-(1-\phi)^{n-1}\right]}$, we have:

$$q_j + p_j \frac{\partial q_j}{\partial p_j} \geq 0 \tag{A–21}$$

Hence, the boundary condition is $v_2 \leq v \leq v_3$. Therefore, $p^* = v - t$ is the equilibrium price if $v_2 \leq v \leq v_3$. ■

Claim 4 *There exists a unique corner solution to p_j^* in Region 1 ($t \leq v \leq v_1$).*

Proof. For $p^* = v - \frac{t}{2}$ to be an equilibrium, we need to show that any deviation is unprofitable, i.e., $q_j + p_j \frac{\partial q_j}{\partial p_j} = q_j + \left(v - \frac{t}{2}\right) \frac{\partial q_j}{\partial p_j} \leq 0$ for p_j slightly higher than $v - \frac{t}{2}$; and $q_j + p_j \frac{\partial q_j}{\partial p_j} = q_j + \left(v - \frac{t}{2}\right) \frac{\partial q_j}{\partial p_j} \geq 0$ for p_j slightly lower than $v - \frac{t}{2}$.

$$q_j = \begin{cases} \frac{2\phi}{N} \frac{v - p_j}{t} & \text{for} \quad p_j \text{ slightly above } v - \frac{t}{2} \\ \frac{1}{N}\phi\left[1 - (1-\phi)^{n-1}\right]\left(1 + \frac{p_k - p_j}{t}\right) + \frac{1}{N}\phi(1-\phi)^{n-1} & \\ + \frac{2}{N}\left\{ \begin{array}{c} \frac{N-n}{n}\left[1 - (1-\phi)^n\right] \\ + (1-\phi)\left[1 - (1-\phi)^{n-1}\right] \end{array} \right\}\left(\frac{v - p_j}{t} - \frac{1}{2}\right) & \text{for} \quad p_j \text{ slightly below } v - \frac{t}{2} \end{cases} \tag{A–22}$$

When p_j is slightly higher than $v - \frac{t}{2}$,

$$q_j + p_j \frac{\partial q_j}{\partial p_j} = \frac{\phi}{N} - \left(v - \frac{t}{2}\right) \frac{2\phi}{N} \leq 0 \tag{A–23}$$

for $v \geq t$.

When p_j is slightly lower than $v - \frac{t}{2}$,

$$\begin{aligned} q_j + p_j \frac{\partial q_j}{\partial p_j} &= \frac{1}{n}\left[1 - (1-\phi)^n\right] - \left(v - \frac{t}{2}\right) \frac{1}{N} \left\{ \begin{array}{c} (2-\phi)\left[1 - (1-\phi)^{n-1}\right] \\ +2\frac{N-n}{n}\left[1 - (1-\phi)^n\right] \end{array} \right\} \\ &\geq 0 \end{aligned} \tag{A–24}$$

for $v \leq v_1 \equiv \frac{t}{2} + \frac{Nt[1-(1-\phi)^n]}{n\left\{(2-\phi)\left[1-(1-\phi)^{n-1}\right] + \frac{2}{n}(N-n)[1-(1-\phi)^n]\right\}}$.

In sum, the boundary condition for $p^* = v - \frac{t}{2}$ to be an equilibrium is $t \leq v \leq v_1$. Therefore, $p^* = v - \frac{t}{2}$ is the equilibrium price if $t \leq v \leq v_1$. ■

Claim 5 *The exists a unique interior solution to p_j^* in Region 2 ($v_1 < v < v_2$).*

Proof. This follows straightforwardly from the continuity of v. In other words, $p^* = \frac{N(2v-t)[1-(1-\phi)^n] - 2n\phi(v-t)}{4N[1-(1-\phi)^n] - n\phi(1-\phi)^{n-1} - 3n\phi}$ is the equilibrium price when $v_1 < v < v_2$. ■

Equilibrium Price. From the preceding claims, it follows that in the symmetric pure strategy Nash equilibrium, firm j's optimal pricing strategy is:

$$p^* = \begin{cases} v - \frac{t}{2} & \text{for} \quad t \leq v \leq v_1 \\ \frac{N(2v-t)[1-(1-\phi)^n] - 2n\phi(v-t)}{4N[1-(1-\phi)^n] - n\phi(1-\phi)^{n-1} - 3n\phi} & \text{for} \quad v_1 < v < v_2 \\ v - t & \text{for} \quad v_2 \leq v \leq v_3 \\ \frac{Nt[1-(1-\phi)^n]}{n\phi\left[1-(1-\phi)^{n-1}\right]} & \text{for} \quad v_3 < v \leq v_4 \end{cases} \tag{A–25}$$

where

$$
\begin{aligned}
v_1 &= \frac{t}{2} + \frac{Nt\left[1-(1-\phi)^n\right]}{n\left\{(2-\phi)\left[1-(1-\phi)^{n-1}\right] + \frac{2}{n}(N-n)\left[1-(1-\phi)^n\right]\right\}} \\
v_2 &= t + \frac{Nt\left[1-(1-\phi)^n\right]}{n\left\{(2-\phi)\left[1-(1-\phi)^{n-1}\right] + \frac{2}{n}(N-n)\left[1-(1-\phi)^n\right]\right\}} \\
v_3 &= t + \frac{Nt\left[1-(1-\phi)^n\right]}{n\phi\left[1-(1-\phi)^{n-1}\right]} \\
v_4 &= t + \frac{N^2 t\left[1-(1-\phi)^n\right]^2}{n\phi\left\{N\left[1-(1-\phi)^n\right] - n\phi\left[1-(1-\phi)^{n-1}\right]\right\}\left[1-(1-\phi)^{n-1}\right]}
\end{aligned}
\tag{A–26}
$$

Derivation of Equilibrium Revenue

The equilibrium revenue is obtained by multiplying the equilibrium price given in equation (A–25) by the demand given in equation (A–6), noting that the marginal cost of production is zero. The equilibrium profit of firm j is

$$\pi^* = R^* - A(\phi;\alpha) \tag{A–27}$$

In the symmetric pure strategy Nash equilibrium, firm j's revenue is given by

$$
R^* = \begin{cases}
\left(v-\frac{t}{2}\right)\frac{\phi}{N} & \text{for} \quad t \le v \le v_1 \\
\frac{\{N(2v-t)[1-(1-\phi)^n]-2n\phi(v-t)\}^2\left\{2N[1-(1-\phi)^n]-n\phi\left[1+(1-\phi)^{n-1}\right]\right\}}{Nnt\left\{4N[1-(1-\phi)^n]-n\phi\left[3+(1-\phi)^{n-1}\right]\right\}^2} & \text{for} \quad v_1 < v < v_2 \\
\frac{v-t}{Nn}\left\{N\left[1-(1-\phi)^n\right]-n\phi\left[1-(1-\phi)^{n-1}\right]\right\} & \text{for} \quad v_2 \le v \le v_3 \\
\frac{Nt[1-(1-\phi)^n]^2}{n^2\phi\left[1-(1-\phi)^{n-1}\right]} & \text{for} \quad v_3 < v \le v_4
\end{cases}
\tag{A–28}
$$

Examination of the Case When $N = n = 2$

When $n = 2$ and $N = 2$ the spokes model reduces to the classic Hotelling model, and this gives us an opportunity to compare the results obtained using our formulation with that of Soberman (2004). It suffices to show that the demand function in Soberman (2004) is identical to ours when $N = n = 2$. If that is true, everything else must follow.

The demand functions in Soberman (2004) (equations 1 and 2, p. 1746) are as follows:

$$
\begin{aligned}
q_1 &= \phi_1(1-\phi_2)x_1 + \phi_1\phi_2 y_1 \\
q_2 &= \phi_2(1-\phi_1)x_2 + \phi_1\phi_2 y_2
\end{aligned}
\tag{A–29}
$$

where

$$x_i = \begin{cases} \frac{v-p_j}{t} & \text{for } \frac{v-p_j}{t} < 1 \\ 1 & \text{for } \frac{v-p_j}{t} \geq 1 \end{cases}$$

$$y_i = \frac{1}{2} + \frac{p_j - p_i}{2t}, i \neq j, i, j = 1, 2. \quad \text{(A–30)}$$

Substituting x_i and y_i into q_i, we have

$$q_i = \begin{cases} \phi_i \left(1 - \phi_j\right) \left(\frac{v-p_j}{t}\right) + \phi_i \phi_j \left(\frac{1}{2} + \frac{p_j - p_i}{2t}\right) & \text{for } \frac{v-p_j}{t} < 1 \\ \phi_i \left(1 - \phi_j\right) + \phi_i \phi_j \left(\frac{1}{2} + \frac{p_j - p_i}{2t}\right) & \text{for } \frac{v-p_j}{t} \geq 1 \end{cases} \quad \text{(A–31)}$$

It is straightforward to see that when $N = n = 2$, equation (6) in the paper becomes

$$q_j = \begin{cases} \phi \left(1 - \widehat{\phi}\right) \frac{v-p_j}{t} + \phi\widehat{\phi} \left(\frac{1}{2} + \frac{p_k - p_j}{2t}\right) & \text{for } \frac{1}{2} < \frac{v-p_j}{t} < 1 \\ \phi \left(1 - \widehat{\phi}\right) + \phi\widehat{\phi} \left(\frac{1}{2} + \frac{p_k - p_j}{2t}\right) & \text{for } \frac{v-p_j}{t} \geq 1 \end{cases} \quad \text{(A–32)}$$

Clearly, equations (A–31) and (A–32) are identical.

On the Joint Probability of Considering Both Brands j and k by a Consumer in Group 1

In group 1, the probability that a consumer includes brands j and k in her consideration set, where brand j is the local brand for a consumer located on spoke l_j and brand k is a nonlocal product is given by:

$$\text{prob}\,[j,k] \equiv \frac{1}{n-1}\phi\left[1 - (1-\phi)^{n-1}\right]. \quad \text{(A–33)}$$

We first provide a few numerical examples and then prove the general case.

When $n = 3$ and $\phi = \frac{1}{3}$, prob$[j,k] = \frac{5}{54}$. Suppose there are three spokes l_j, l_k, l_m and the consumer is on l_j. As the consideration set size is two and the consumer only includes one nonlocal brand in her consideration set, the probability that brand k is included in her consideration set (prob$[k]$) is the sum of two probabilities: the probability that brand k is the only nonlocal brand that the consumer is aware of (prob$[k$ only$]$), and the probability that the consumer is aware of both brands k and m (prob$[k$ and $m]$) but chooses k as the second preferred brand. Now prob$[k$ only$] = \frac{1}{3} \times \left(1 - \frac{1}{3}\right) = \frac{2}{9}$. Next prob$[k$ and $m] = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$, and note that it is equally likely that the consumer includes brand k or m in her consideration. Therefore, it follows that prob$[k] = \frac{2}{9} + \frac{1}{2} \times \frac{1}{9} = \frac{5}{18}$ and prob$[j,k] = \frac{1}{3} \times \frac{5}{18} = \frac{5}{54}$.

When $n = 4$ *and* $\phi = \frac{1}{2}$, *prob*$[j,k] = \frac{7}{48}$. Suppose there are four spokes l_j, l_k, l_m, l_n and the consumer is on l_j. The probability that brand k is in her consideration set (prob$[k]$) is the sum of four probabilities: the probability that brand k is the only nonlocal brand that the consumer is aware of (prob$[k$ only$]$), the probability that the consumer is aware of both brands k and m (prob$[k$ and $m]$) but chooses k as the second preferred brand, the probability that the consumer is aware of both brands k and n (prob$[k$ and $n]$) but chooses k as the second preferred brand, and the probability that the consumer is aware of brands k, m and n (prob$[k,m,n]$) but chooses k as the second preferred brand. It is easy to compute prob$[k \text{ only}] = \frac{1}{2} \times \left(1 - \frac{1}{2}\right)^2 = \frac{1}{8}$; prob$[k \text{ and } m] = prob\,[k \text{ and } n] = \left(\frac{1}{2}\right)^2 \times \left(1 - \frac{1}{2}\right) = \frac{1}{8}$; prob$[k,m,n] = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$. It follows that prob$[k] = \frac{1}{8} + \frac{1}{2} \times \frac{1}{8} + \frac{1}{2} \times \frac{1}{8} + \frac{1}{3} \times \frac{1}{8} = \frac{7}{24}$ and prob$[j,k] = \frac{1}{2} \times \frac{7}{24} = \frac{7}{48}$.

When $n = 4$ *and* $\phi = \frac{1}{3}$, *prob*$[j,k] = \frac{19}{243}$. prob$[k \text{ only}] = \frac{1}{3} \times \left(1 - \frac{1}{3}\right)^2 = \frac{4}{27}$; prob$[k \text{ and } m] = prob\,[k \text{ and } n] = \left(\frac{1}{3}\right)^2 \times \left(1 - \frac{1}{3}\right) = \frac{2}{27}$; prob$[k,m,n] = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$. It follows that prob$[k] = \frac{4}{27} + \frac{1}{2} \times \frac{2}{27} + \frac{1}{2} \times \frac{2}{27} + \frac{1}{3} \times \frac{1}{27} = \frac{19}{81}$ and prob$[j,k] = \frac{1}{3} \times \frac{19}{81} = \frac{19}{243}$.

More generally, suppose there are n spokes, $l_1, l_2, \ldots, l_n$ and focus attention on a consumer located in spoke l_j. Given that the consideration set size is two and that the consumer on l_j chooses randomly her nonlocal brand from the set of products she is informed, the probability that brand k is included in her consideration set (prob$[k]$) is the sum of the following probabilities:

$$
\begin{aligned}
\text{prob}\,[k \text{ only}] &= \phi\,(1-\phi)^{n-2}\,, \\
\frac{(n-2)!}{(n-3)!2!}\text{prob}\,[k \text{ and } m] &= \frac{1}{2}\,(n-2)\,\phi^2\,(1-\phi)^{n-3}\,, \\
\frac{(n-2)!}{(n-4)!3!}\text{prob}\,[k,m,n] &= \frac{1}{6}\,(n-2)\,(n-3)\,\phi^3\,(1-\phi)^{n-4}\,, \qquad \text{(A–34)} \\
&\vdots \\
\frac{1}{n-1}\text{prob}\,[1,\ldots,j-1,j+1,\ldots,n] &= \frac{1}{n-1}\phi^{n-1}.
\end{aligned}
$$

By binomial theorem, we know that

$$\begin{aligned}(\phi+1-\phi)^{n-1} &= \sum_{i=0}^{n-1}\binom{n-1}{i}\phi^{n-i-1}(1-\phi)^{i}, \\ \frac{1}{n-1}(\phi+1-\phi)^{n-1} &= \sum_{i=0}^{n-1}\frac{(n-2)!}{i!\,(n-i-1)!}\phi^{n-i-1}(1-\phi)^{i}. \end{aligned} \quad \text{(A–35)}$$

It follows that the probabilities in equation (A–34) sum up to $\frac{1}{n-1}\left[1-(1-\phi)^{n-1}\right]$. Therefore,

$$\text{prob}\,[j,k] = \frac{1}{n-1}\phi\left[1-(1-\phi)^{n-1}\right]. \quad \text{(A–36)}$$

On Quadratic Transportation Costs

In developing the model, we made a few simplifying assumptions. We now assess the robustness of our model predictions to some of these assumptions. In our model we assumed the cost of any mismatch between a consumer's taste and product's feature was linear. We can relax this assumption and allow for quadratic costs as follows. If the consumer located at (l_j, x) is aware of her local brand j and decides to buy the product, then the indirect utility derived by her is given by:

$$U(l_j, x, p_j) = v - tx^2 - p_j, \quad \text{(A–37)}$$

However, if the consumer purchases any other product of which she is informed, namely variety k such that $k \neq j$, then the indirect utility obtained from this nonlocal brand will be given by:

$$U(l_j, x, p_k) = v - t(1-x)^2 - p_k. \quad \text{(A–38)}$$

It is straightforward to show that the marginal consumer who is indifferent between the two products is located at a distance $\frac{1}{2} + \frac{p_k - p_j}{2t}$ from the local brand j, and the demand for brand j is $\min\{\frac{1}{2} + \frac{p_k - p_j}{2t}, 1\}$. More generally, the demand from each of the four groups of consumers is as follows.

Group 1. The demand from this group of consumers for firm j's product is:

$$\frac{2}{N}\frac{1}{n-1}\phi\left[1-(1-\phi)^{n-1}\right]\sum_{k\neq j, k\in\{1,\ldots,n\}}\max\left\{\min\left\{\frac{1}{2}+\frac{p_k-p_j}{2t},1\right\},0\right\}. \quad \text{(A–39)}$$

Group 2. Firm j's demand from this group of consumers is given by:

$$\frac{2}{N}\phi\left(1-\phi\right)^{n-1}\min\left\{\max\left\{0,\sqrt{\frac{v-p_j}{t}}\right\},\frac{1}{2}\right\}. \tag{A–40}$$

Group 3. The demand from this segment for firm j's product is:

$$\frac{2}{N}\frac{N-n}{n}\left[1-\left(1-\phi\right)^n\right]\min\left\{\max\left\{0,\sqrt{\frac{v-p_j}{t}}-\frac{1}{2}\right\},\frac{1}{2}\right\}. \tag{A–41}$$

Group 4. Firm j's demand from this segment is given by:

$$\frac{2}{N}\frac{1}{n-1}\left(1-\phi\right)\left[1-\left(1-\phi\right)^{n-1}\right]\sum_{k\neq j,k\in\{1,\ldots,n\}}\min\left\{\max\left\{0,\sqrt{\frac{v-p_j}{t}}-\frac{1}{2}\right\},\frac{1}{2}\right\}. \tag{A–42}$$

We assume that $\sqrt{\frac{v-p_j}{t}}>\frac{1}{2}$ and $\sqrt{\frac{|p_k-p_j|}{t}}\leq 1$ so that there is some competition between the local and nonlocal brands. Note that when $\frac{1}{2}<\sqrt{\frac{v-p_j}{t}}<1$ all consumers derive a positive surplus on purchasing their local brand, but for some consumers the base valuation is not high enough to obtain a positive surplus by purchasing a nonlocal brand. By contrast, when $\sqrt{\frac{v-p_j}{t}}\geq 1$, every consumer gains a surplus from purchasing the local or the nonlocal brand. As firms are symmetric the demand from Group 1 (equation A–39) and Group 2 (equation A–40) can be simplified to $\frac{1}{N}\phi\left[1-\left(1-\phi\right)^{n-1}\right]\left(1+\frac{p_k-p_j}{t}\right)$ and $\frac{1}{N}\phi\left(1-\phi\right)^{n-1}$, respectively, when $\frac{1}{2}<\sqrt{\frac{v-p_j}{t}}<1$. After simplifying the demand from Groups 3 and 4 along similar lines, we obtain the total demand for firm j's product, which is given by:

$$q_j=\begin{cases}\begin{array}{l}\frac{1}{N}\phi\left[1-\left(1-\phi\right)^{n-1}\right]\left(1+\frac{p_k-p_j}{t}\right)+\frac{1}{N}\phi\left(1-\phi\right)^{n-1}\\ +\frac{2}{N}\left\{\begin{array}{c}\frac{N-n}{n}\left[1-\left(1-\phi\right)^n\right]\\ +\left(1-\phi\right)\left[1-\left(1-\phi\right)^{n-1}\right]\end{array}\right\}\left(\sqrt{\frac{v-p_j}{t}}-\frac{1}{2}\right)\end{array} & \text{for } \frac{1}{2}<\sqrt{\frac{v-p_j}{t}}<1\\ \begin{array}{l}\frac{1}{N}\phi\left[1-\left(1-\phi\right)^{n-1}\right]\left(1+\frac{p_k-p_j}{t}\right)+\frac{1}{N}\phi\left(1-\phi\right)^{n-1}\\ +\frac{1}{N}\left\{\frac{N-n}{n}\left[1-\left(1-\phi\right)^n\right]+\left(1-\phi\right)\left[1-\left(1-\phi\right)^{n-1}\right]\right\}\end{array} & \text{for } \sqrt{\frac{v-p_j}{t}}\geq 1\end{cases} \tag{A–43}$$

It is easy to see that the demand for firm j when $\sqrt{\frac{v-p_j}{t}}\geq 1$ is identical to that when transportation cost is linear and when $\frac{v-p_j}{t}\geq 1$. Therefore, any result related to Region 4 must hold. We henceforth focus on Region 2, where the demand differs under quadratic transportation cost. Clearly, there is no closed-form solution when $\frac{1}{2}<\sqrt{\frac{v-p_j}{t}}<1$ due to the fact that the term $\sqrt{\frac{v-p_j}{t}}$ introduces high-degree polynomials. We have to resort to numerical analysis to examine the impact of quadratic transportation cost for Region 2. We first study the effect advertising reach (ϕ) on prices in Region 2 (part i of Proposition 1). Consider a baseline case where $t = 1$, $k = \frac{n}{N} = 0.3$, $\phi = 0.3$, $v = 1.5$, and $n = 10$. In this baseline case the equilibrium price $p_j^* = 0.717$. Now holding everything else constant, if we just increase ϕ to 0.5, the corresponding $p_j^* = 0.72$; if we increase ϕ further to 0.8, then p_j^* becomes 0.733. Thus equilibrium price increases with advertising reach, a finding qualitatively consistent with part i of Proposition 1. Next, let us turn our attention to the effect of diversity in consumer tastes (N) on prices in Region 2 (see part i of Proposition 2). Now in the earlier baseline case, if we raise $k = \frac{n}{N}$ to 0.6 holding everything else constant (thus N is smaller given n), the corresponding p_j^* increases to 0.724. If we further raise k to 0.9, the corresponding $p_j^* = 0.737$. This pattern of results is qualitatively consistent with part i of Proposition 2. Therefore, we conclude that the original results obtained with linear transportation cost are generalizable to quadratic transportation cost.

An Alternative Demand Setup

In our model, we assumed that consumers considered at most two products and the consideration set was comprised of the local brand and a nonlocal brand. Furthermore, consumers were equally likely to consider purchasing any of the available nonlocal brands they were aware of. We now consider the situation where consumers prefer their local brand (if available) and an *à priori* randomly chosen nonlocal brand.

Segment 1. A consumer in this group is aware of her local brand as well as the nonlocal brand in her consideration set, and both products are available in the market. For any consumer located on l_j or l_k, j, $k \in \{1, \ldots, n\}$, both brand j and brand k are her desired brands with conditional probability $\frac{1}{N-1}$, and with probability ϕ^2 these consumers are informed of both brands. The density of such consumers is $\frac{2}{N}$, and the mass of the consumers served by firm j is:

$$\frac{2}{N}\frac{1}{N-1}\phi^2 \sum_{k \neq j, k \in \{1,\ldots,n\}} \max\left\{\min\left\{\frac{1}{2} + \frac{p_k - p_j}{2t}, 1\right\}, 0\right\}. \tag{A–44}$$

Segment 2. The consumers in this group are aware of their local brands, but their second preferred brand is either not available or they are uninformed of the nonlocal brand in their consideration sets, though the product is available in the marketplace. For any consumer on l_j, with conditional probability $\frac{N-n+1}{N-1}$ her second preferred brand is not available; and with conditional probability $\frac{1}{N-1}(1-\phi)$, she is uninformed of her second preferred brand, although it is available. As the density of such consumers is $\frac{2}{N}$, firm j's demand from this group of consumers is given by:

$$\frac{2}{N}\phi\left[\frac{N-n+1}{N-1}+\frac{1}{N-1}(1-\phi)\sum_{k\neq j,k\in\{1,\ldots,n\}}\right]\min\left\{\max\left\{0,\frac{v-p_j}{t}\right\},\frac{1}{2}\right\}. \tag{A–45}$$

Next we consider consumers for whom brand j is the nonlocal product.

Segment 3. The local brand preferred by consumers in this group is not produced by any of the n firms, and product j is the nonlocal brand in their consideration sets. For any consumer on l_i, $i\neq j$, $i\notin\{1,\ldots,n\}$, brand j is her second preferred brand with conditional probability $\frac{1}{N-1}$, the density of such consumers is $\frac{2}{N}(N-n)$ and, therefore, the demand from this segment for firm j's product is:

$$\frac{2}{N}\frac{N-n}{N-1}\phi\min\left\{\max\left\{0,\frac{v-p_j}{t}-\frac{1}{2}\right\},\frac{1}{2}\right\}. \tag{A–46}$$

Segment 4. Consumers in this segment are uninformed of their local brand and product j is the nonlocal brand in their consideration set. For any consumer on l_k, $k\neq j$, $k\in\{1,\ldots,n\}$, variety j is her second preferred brand with conditional probability $\frac{1}{N-1}$; and with probability $\phi(1-\phi)$, she is uninformed of her first preferred brand (k) but is informed of her second preferred brand (j); the density of such consumers is $\frac{2}{N}$. Firm j's demand from this segment is given by:

$$\frac{2}{N}\frac{1}{N-1}\phi(1-\phi)\sum_{k\neq j,k\in\{1,\ldots,n\}}\min\left\{\max\left\{0,\frac{v-p_j}{t}-\frac{1}{2}\right\},\frac{1}{2}\right\}. \tag{A–47}$$

We assume that $\frac{v-p_j}{t}>\frac{1}{2}$ and $\frac{|p_k-p_j|}{t}\leq 1$ so that there is some competition between the local and nonlocal brands. We obtain the following demand for firm j's product:

$$q_j=\begin{cases}\begin{array}{c}\frac{n-1}{N(N-1)}\phi^2\left(1+\frac{p_k-p_j}{t}\right)+\frac{1}{N}\phi\left[\frac{N-n+1}{N-1}+\frac{n-1}{N-1}(1-\phi)\right]\\+\frac{2}{N}\left[\frac{N-n}{N-1}\phi+\frac{n-1}{N-1}\phi(1-\phi)\right]\left(\frac{v-p_j}{t}-\frac{1}{2}\right)\end{array} & \text{for } \frac{1}{2}<\frac{v-p_j}{t}<1\\ \begin{array}{c}\frac{n-1}{N(N-1)}\phi^2\left(1+\frac{p_k-p_j}{t}\right)+\frac{1}{N}\phi\left[\frac{N-n+1}{N-1}+\frac{n-1}{N-1}(1-\phi)\right]\\+\frac{1}{N}\left[\frac{N-n}{N-1}\phi+\frac{n-1}{N-1}\phi(1-\phi)\right]\end{array} & \text{for } \frac{v-p_j}{t}\geq 1\end{cases} \tag{A–48}$$

It can be shown that

$$p_j^* = \begin{cases} \frac{(t-2v)(1-\phi)+2v(N-n\phi)+nt\phi}{4N-3n\phi-4+3\phi} & \text{for Region 2} \\ \frac{t[2N-n\phi-(1-\phi)]}{\phi(n-1)} & \text{for Region 4} \end{cases} \tag{A–49}$$

Recall that in Region 2, $v_1 < v < v_2$ which implies $v \in (t, 2t)$; in Region 4, $v_3 < v \leq v_4$ which implies $v > 2t$. We therefore have

$$\frac{\partial p_j^*}{\partial \phi} = \begin{cases} (n-1)\frac{2N(2t-v)+2v-t}{(4N+3\phi-3n\phi-4)^2} > 0 & \text{for Region 2} \\ -\frac{t}{\phi^2}\frac{2N-1}{n-1} < 0 & \text{for Region 4} \end{cases} \tag{A–50}$$

$$\frac{\partial p_j^*}{\partial N} = \begin{cases} -\frac{2[2t(1-\phi)+v\phi+n\phi(2t-v)]}{(4N+3\phi-3n\phi-4)^2} < 0 & \text{for Region 2} \\ \frac{2t}{\phi(n-1)} > 0 & \text{for Region 4} \end{cases} \tag{A–51}$$

Note that the comparative statics in equations (A–50) and (A–51) are consistent with Propositions 1 and 2.

APPENDIX B: PROOFS

Proof of Proposition 1

Proof. It is straightforward to see that $\frac{\partial p^*}{\partial \phi} = 0$ when $t \leq v \leq v_1$ and $v_2 \leq v \leq v_3$. When $v_1 < v < v_2$,

$$\frac{\partial p^*}{\partial \phi} = \frac{\partial}{\partial \phi}\left\{\frac{N(2v-t)[1-(1-\phi)^n] - 2n\phi(v-t)}{4N[1-(1-\phi)^n] - n\phi(1-\phi)^{n-1} - 3n\phi}\right\} \tag{B–1}$$

$$= -\frac{n}{(1-\phi)^2\left\{4N[1-(1-\phi)^n] - 3n\phi - n\phi(1-\phi)^{n-1}\right\}^2} \cdot \left\{\begin{array}{c} N(1-\phi)^2(2v-5t) + N(1-\phi)^{2n}(2v-t) \\ +(1-\phi)^n\left[\begin{array}{c} 2N((3t-2v)+\phi(2nt+2v-5t)) \\ -\phi^2(n-1)(5Nt-2nt-2v(N-n)) \end{array}\right] \end{array}\right\}$$

In a monopolistic market, the number of firms (n) is large. Following Grossman and Shapiro (1984), we use the approximation $(1-\phi)^n \approx 0$. With this approximation, equation (B–1) can be simplified to $\frac{\partial p^*}{\partial \phi} = \frac{Nn(5t-2v)}{(4N-3n\phi)^2}$. Clearly, the sign of $\frac{\partial p^*}{\partial \phi}$ depends on consumer valuation v. We now consider the limiting conditions of v_i to define the range of valuations that v_i, $i = 1 \ldots 4$ can take.

$$\begin{aligned}
\lim_{\phi \to 0} v_1 &= \frac{t}{2} + \lim_{\phi \to 0} \frac{Nt\left[1-(1-\phi)^n\right]}{n\left\{(2-\phi)\left[1-(1-\phi)^{n-1}\right] + \frac{2}{n}(N-n)\left[1-(1-\phi)^n\right]\right\}} \\
&= \frac{t}{2} + \frac{Nt}{2(N-1)} \\
\lim_{\phi \to 1} v_1 &= \frac{t}{2} + \frac{Nt}{2N-n} \\
\lim_{\phi \to 0} v_2 &= t + \lim_{\phi \to 0} \frac{Nt\left[1-(1-\phi)^n\right]}{n\left\{(2-\phi)\left[1-(1-\phi)^{n-1}\right] + \frac{2}{n}(N-n)\left[1-(1-\phi)^n\right]\right\}} \\
&= t + \frac{Nt}{2(N-1)} \\
\lim_{\phi \to 1} v_2 &= t + \frac{Nt}{2N-n} \qquad \text{(B–2)} \\
\lim_{\phi \to 0} v_3 &= t + \lim_{\phi \to 0} \frac{Nt\left[1-(1-\phi)^n\right]}{n\phi\left[1-(1-\phi)^{n-1}\right]} \\
&= \infty \\
\lim_{\phi \to 1} v_3 &= t + \frac{Nt}{n} \\
\lim_{\phi \to 0} v_4 &= t + \lim_{\phi \to 0} \frac{N^2 t\left[1-(1-\phi)^n\right]^2}{n\phi\left\{N\left[1-(1-\phi)^n\right] - n\phi\left[1-(1-\phi)^{n-1}\right]\right\}\left[1-(1-\phi)^{n-1}\right]} \\
&= \infty \\
\lim_{\phi \to 1} v_4 &= t + \frac{N^2 t}{n(N-n)}
\end{aligned}$$

As $N \geq n \geq 2$, it follows that $v_1 \in \left(t, \frac{3t}{2}\right)$, $v_2 \in \left(\frac{3t}{2}, 2t\right)$, $v_3 \in (2t, \infty)$, and $v_4 \in (5t, \infty)$. Hence, when $v_1 < v < v_2$, we obtain $\frac{\partial p^*}{\partial \phi} > 0$. Next, turning our attention to the case when $v_3 < v \leq v_4$, we find that

$$\begin{aligned}
\frac{\partial p^*}{\partial \phi} &= \frac{\partial}{\partial \phi}\left\{\frac{Nt\left[1-(1-\phi)^n\right]}{n\phi\left[1-(1-\phi)^{n-1}\right]}\right\} \\
&= Nt\frac{\left[2(1-\phi) - (n-1)\phi^2 - (1-\phi)^n\right](1-\phi)^n - (1-\phi)^2}{n\phi^2(1-\phi)^2\left[1-(1-\phi)^{n-1}\right]^2} \qquad \text{(B–3)}
\end{aligned}$$

Note that the denominator of $\frac{\partial p^*}{\partial \phi}$ is positive for $\phi \in (0,1)$. It can be shown that the numerator is always negative for any $\phi \in (0,1)$). Therefore, it follows that $\frac{\partial p^*}{\partial \phi} < 0$. ■

Proof of Corollary 1

Proof. It suffices to consider the cases of $n = 2$ and $n = 3$.

Case 1: When $n = 2$,

$$\frac{\partial p^*}{\partial \phi} = \frac{3Nt + 2v - 2Nv - 2t}{(4N + \phi - 2N\phi - 4)^2} \tag{B–4}$$

Recall that $v_1 \in \left(t, \frac{3t}{2}\right)$ and $v_2 \in \left(\frac{3t}{2}, 2t\right)$. Now suppose $v = \frac{3t}{2}$. Then the numerator of the above expression $3Nt + 2v - 2Nv - 2t = t$ is positive and hence $\frac{\partial p^*}{\partial \phi} > 0$. Next suppose $v = 2t$. Then $3Nt + 2v - 2Nv - 2t = (2 - N)\, t$, implying that the numerator is negative as $N > 2$. Consequently, $\frac{\partial p^*}{\partial \phi} \leq 0$ as the denominator is always positive.

Case 2: When $n = 3$,

$$\frac{\partial p^*}{\partial \phi} = 3\frac{12(v-t)(1-\phi) + Nt\left(\phi^2 - 14\phi + 18\right) - 2Nv\left(\phi^2 - 6\phi + 6\right)}{\left(12N + 6\phi - 12N\phi - 3\phi^2 + 4N\phi^2 - 12\right)^2} \tag{B–5}$$

In this case $\frac{\partial p^*}{\partial \phi}$ can be either positive or negative as the numerator can take either sign. On one hand, the numerator can take a positive value as

$$\begin{aligned} & \lim_{\phi \to 0} \left[12(v-t)(1-\phi) + Nt\left(\phi^2 - 14\phi + 18\right) - 2Nv\left(\phi^2 - 6\phi + 6\right)\right] \\ = \; & 6t(2 - 3N) + 12v(N-1) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B–6)} \\ \geq \; & 0 \text{ for } v \geq 6t\frac{3N-2}{12N-12} \end{aligned}$$

On the other hand, the numerator can take a negative value as

$$\begin{aligned} & \lim_{\phi \to 1} \left[12(v-t)(1-\phi) + Nt\left(\phi^2 - 14\phi + 18\right) - 2Nv\left(\phi^2 - 6\phi + 6\right)\right] \\ = \; & N(2v - 5t) < 0 \end{aligned} \tag{B–7}$$

■

Proof of Proposition 2

Proof. It is straightforward to see that $\frac{\partial p^*}{\partial N} = 0$ when $t \leq v \leq v_1$ and $v_2 \leq v \leq v_3$. When $v_1 < v < v_2$,

$$\begin{aligned} \frac{\partial p_j^*}{\partial N} &= \frac{n\phi\left[1 - (1-\phi)^n\right]}{\left\{n\phi\left[3 + (1-\phi)^{n-1}\right] - 4N\left[1 - (1-\phi)^n\right]\right\}^2}\left\{(2v - t)\left[1 - (1-\phi)^{n-1}\right] - 4t\right\} \\ &< 0 \end{aligned} \tag{B–8}$$

Note that as $v_1 < v < v_2$ implies $v \in (t, 2t)$, we have $(2v - t)\left[1 - (1-\phi)^{n-1}\right] \in (t, 3t)$. Consequently, the numerator of the above equation is negative, its denominator is always positive and the comparative statics is negative.

When $v_3 < v \leq v_4$, we have

$$\frac{\partial p_j^*}{\partial N} = \frac{t\left[1-(1-\phi)^n\right]}{n\phi\left[1-(1-\phi)^{n-1}\right]} > 0 \tag{B–9}$$

■

Proof of Proposition 3

Proof. As we are focusing on monopolistic competition, we use the approximation $(1-\phi)^n \approx 0$. When $v_1 < v < v_2$, on solving firm j's profit maximization problem using the demand

$$q_j = \begin{cases} \frac{1}{N}\phi\left[1-\left(1-\widehat{\phi}\right)^{n-1}\right]\left(1+\frac{p_k-p_j}{t}\right)+\frac{1}{N}\phi\left(1-\widehat{\phi}\right)^{n-1} \\ +\frac{2}{N}\left\{\begin{array}{c}\frac{N-n}{n}\left[1-(1-\phi)\left(1-\widehat{\phi}\right)^{n-1}\right]+ \\ \left(1-\widehat{\phi}\right)\left[1-(1-\phi)\left(1-\widehat{\phi}\right)^{n-2}\right]\end{array}\right\}\left(\frac{v-p_j}{t}-\frac{1}{2}\right) & \text{for } \frac{1}{2}<\frac{v-p_j}{t}<1 \\ \frac{1}{N}\phi\left[1-\left(1-\widehat{\phi}\right)^{n-1}\right]\left(1+\frac{p_k-p_j}{t}\right)+\frac{1}{N}\phi\left(1-\widehat{\phi}\right)^{n-1} \\ +\frac{1}{N}\left\{\begin{array}{c}\frac{N-n}{n}\left[1-(1-\phi)\left(1-\widehat{\phi}\right)^{n-1}\right]+ \\ \left(1-\widehat{\phi}\right)\left[1-(1-\phi)\left(1-\widehat{\phi}\right)^{n-2}\right]\end{array}\right\} & \text{for } \frac{v-p_j}{t}\geq 1 \end{cases} \tag{B–10}$$

we obtain

$$p^* = \frac{N(2v-t)+nt\left(\phi+\widehat{\phi}\right)-2nv\widehat{\phi}}{4N+n\phi-4n\widehat{\phi}} \tag{B–11}$$

$$\frac{\partial \pi_j}{\partial \phi} = \frac{1}{N}p_j\left(1+\frac{p_k-p_j}{t}\right)-A_\phi \tag{B–12}$$

Noting that firms are symmetric and simplifying the expression, we have

$$\frac{\partial \pi_j}{\partial \phi} = \frac{1}{N}p^*-A_\phi = 0 \tag{B–13}$$

This implies that $p^* = NA_\phi$. Hence we obtain

$$A_\phi = \frac{1}{N}\frac{N(2v-t)+nt\left(\phi+\widehat{\phi}\right)-2nv\widehat{\phi}}{4N+n\phi-4n\widehat{\phi}} \tag{B–14}$$

As firms are symmetric, it is easy to see that

$$A_\phi = \frac{N(2v-t)-2n\phi(v-t)}{N(4N-3n\phi)} > 0 \tag{B–15}$$

Further, on totally differentiating p^* and noting that $\widehat{\phi} = \phi$ in symmetric equilibrium, we obtain

$$\frac{dp}{d\alpha} = \frac{Nn}{(4N - 3n\phi)^2}(5t - 2v)\frac{d\phi}{d\alpha} \tag{B–16}$$

Now using equation (B–15), we have

$$\frac{dp}{d\alpha} = \frac{N^3 n (A_\phi)^2}{[N(2v - t) - 2n\phi(v - t)]^2}(5t - 2v)\frac{d\phi}{d\alpha} \tag{B–17}$$

Note that $5t - 2v > 0$ because $v_1 < v < v_2$ implies $v \in (t, 2t)$. It follows that $Sgn\left\{\frac{dp}{d\alpha}\right\} = Sgn\{\Phi\}$ (see equation B–24).

When $v_3 < v \leq v_4$, it can be shown that

$$p^* = \frac{Nt + nt\left(\phi - \widehat{\phi}\right)}{n\phi} \tag{B–18}$$

$$\frac{\partial \pi_j}{\partial \phi} = \frac{1}{N}p_j\left(1 + \frac{p_k - p_j}{t}\right) - A_\phi \tag{B–19}$$

By symmetry, $\frac{\partial \pi_j}{\partial \phi} = \frac{1}{N}p^* - A_\phi = 0$. This implies that $p^* = NA_\phi$. Consequently, $A_\phi = \frac{Nt + nt\left(\phi - \widehat{\phi}\right)}{Nn\phi}$. As firms are symmetric, $A_\phi = \frac{t}{n\phi}$. On totally differentiating p^* and noting that $\widehat{\phi} = \phi$ in the symmetric equilibrium, we obtain

$$\frac{dp}{d\alpha} = -\frac{Nt}{n\phi^2}\frac{d\phi}{d\alpha} = -\frac{N}{\phi}A_\phi\frac{d\phi}{d\alpha} > 0 \tag{B–20}$$

which follows from equation (B–26). ■

Proof of Corollary 2

Proof. When $v_1 < v < v_2$, totally differentiating equation (B–15) we obtain

$$[N(4N - 3n\phi)A_{\phi\phi} - 3NnA_\phi + 2n(v - t)]\,d\phi + N(4N - 3n\phi)A_{\phi\alpha}d\alpha = 0 \tag{B–21}$$

Therefore, we have

$$\frac{d\phi}{d\alpha} = -\frac{N(4N - 3n\phi)A_{\phi\alpha}}{N(4N - 3n\phi)A_{\phi\phi} + 2n(v - t) - 3NnA_\phi} \tag{B–22}$$

Note that the numerator is positive. Hence

$$Sgn\left\{\frac{d\phi}{d\alpha}\right\} = Sgn\{3NnA_\phi - N(4N - 3n\phi)A_{\phi\phi} - 2n(v - t)\} \equiv Sgn\{\Phi\} \tag{B–23}$$

Furthermore,

$$Sgn\left\{\Phi\right\} = \begin{cases} + & \text{if} \quad A_\phi > \frac{4N-3n\phi}{3n}A_{\phi\phi} + \frac{2(v-t)}{3N} \\ - & \text{if} \quad A_\phi < \frac{4N-3n\phi}{3n}A_{\phi\phi} + \frac{2(v-t)}{3N} \end{cases} \tag{B–24}$$

Recall that when $v_3 < v \leq v_4$, $A_\phi = \frac{t}{n\phi}$. Totally differentiating this expression we obtain

$$\left(A_\phi + \phi A_{\phi\phi}\right) d\phi + \phi A_{\phi\alpha} d\alpha = 0 \tag{B–25}$$

Therefore, we have

$$\frac{d\phi}{d\alpha} = -\frac{\phi A_{\phi\alpha}}{A_\phi + \phi A_{\phi\phi}} < 0 \tag{B–26}$$

■

Proof of Proposition 4

Proof. When $v_1 < v < v_2$,

$$\begin{aligned} \frac{d\pi}{d\alpha} &= p\frac{dq}{d\alpha} + q\frac{dp}{d\alpha} - A_\phi \frac{d\phi}{d\alpha} - A_\alpha \\ &= q\frac{dp}{d\alpha} - A_\phi \frac{d\phi}{d\alpha} - A_\alpha \end{aligned} \tag{B–27}$$

Since the signs of $\frac{dp}{d\alpha}$ and $\frac{d\phi}{d\alpha}$ are both ambiguous, as shown earlier, it follows that the sign of $\frac{d\pi}{d\alpha}$ is also ambiguous.

When $v_3 < v \leq v_4$,

$$\begin{aligned} \frac{d\pi}{d\alpha} &= p\frac{dq}{d\alpha} + q\frac{dp}{d\alpha} - A_\phi \frac{d\phi}{d\alpha} - A_\alpha \\ &= q\frac{dp}{d\alpha} - A_\phi \frac{d\phi}{d\alpha} - A_\alpha \\ &= \left(\frac{N}{\phi} q A_\phi + 1\right) \frac{\phi A_\phi A_{\phi\alpha}}{A_\phi + \phi A_{\phi\phi}} - A_\alpha \end{aligned} \tag{B–28}$$

Since $\frac{dp}{d\alpha} > 0$ (see equation B–20), $\frac{d\phi}{d\alpha} < 0$ (see equation B–26), and A_ϕ, $A_\alpha \geq 0$, it follows that $\frac{d\pi}{d\alpha} > 0$, which is consistent with Grossman and Shapiro (1984). ■