# THE IMPORTANCE OF MODELING TEMPORAL DEPENDENCE OF TIMING AND QUANTITY IN DIRECT MARKETING

Lichung Jen, Chien-Heng Chou, and Greg M. Allenby

## WEB APPENDIX

*THE GIBBS SAMPLER FOR CONDITIONAL NORMAL HIERARCHICAL BAYES MODEL OF PURCHASE QUANTITY AND INTERPURCHASE TIME*

Denote $y_{ij}$ as the $j^{th}$ log purchase amount of the $i^{th}$ customer, and $t_{ij}$ as the $j^{th}$ log interpurchase time of the $i^{th}$ customer, where i=1,2,…, N and j=1,2,…, $n_i$. Assume that the conditional distribution of $y_{ij}$ given $t_{ij}$ is normally distributed as

$$f\left(y_{ij}\left|t_{ij}\right.\right) \sim N\left(\mu_i + t_{ij}\eta_i,\, \sigma_i^2\right),$$

and that $t_{ij}$ is also normally distributed as

$$f\left(t_{ij}\right) \sim N\left(\upsilon_i,\, \varsigma_i^2\right).$$

Then, the joint distribution of $Y_{ij}$ and $t_{ij}$ is

$$f\left(y_{ij}, t_{ij}\right) \sim N\left(\begin{bmatrix} \mu_i + \upsilon_i\eta_i \\ \upsilon_i \end{bmatrix}, \begin{bmatrix} \eta_i^2\varsigma_i^2 + \sigma_i^2 & \eta_i\varsigma_i^2 \\ \eta_i\varsigma_i^2 & \varsigma_i^2 \end{bmatrix}\right).$$

The prior and the conditional posterior distributions of all parameters are summarized below.

1. The conditional posterior distribution of $\mu_i \mid y_{ij}, t_{ij}, x_i, \beta, \sigma_i, \eta_i, \zeta$ is

$$\text{Normal}\left[\left(\frac{n_i}{2\sigma_i^2}+\frac{1}{2\varsigma^2}\right)^{-1}\left(\frac{\sum_{j=1}^{n_i}\left(y_{ij}-t_{ij}\eta_i\right)}{\sigma_i^2}+\frac{\left(x_i'\beta\right)}{\varsigma^2}\right)\quad,\quad\left(\frac{n_i}{\sigma_i^2}+\frac{1}{\varsigma^2}\right)^{-1}\right].$$

The prior of $\mu_i$ is:

Normal( $x_i'\beta,\ \varsigma^2$ ).

1.1 The conditional posterior distribution of $\beta \mid \mu_i, x_i, \zeta, b_0, V_{b_0}$ is

$$\text{Normal}\left(\left(V_{b_0}^{-1}+\zeta^{-2}\sum x_i x_i'\right)^{-1}\left(V_{b_0}^{-1}b_0+\zeta^{-2}\sum \mu_i x_i'\right),\quad \left(V_{b_0}^{-1}+\zeta^{-2}\sum x_i x_i'\right)\right).$$

The prior of $\beta$ is

Normal($b_0, V_{b_0}$),

where $b_0$=4 and

$$V_{b_0}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}.$$

1.2 The conditional posterior distribution of $\zeta^2$| $\mu_i$, $x_i$, $\beta$, $s_0$, $S_0$ is

$$\text{Inverse Gamma}\left(s_0+\frac{N}{2}\quad,\quad\left(\frac{1}{2}\sum_{i=1}^{N}\left(\mu_i-x_i'\beta\right)^2+S_0^{-1}\right)^{-1}\right).$$

The prior of $\zeta^2$ is Inverse Gamma($s_0$, $S_0$), where $s_0$=2 and $S_0$=0.25.

2. The conditional posterior distribution of $\eta_i \mid y_{ij}, t_{ij}, \mu_i, \sigma_i, \varphi, \varpi$ is

$$\text{Normal}\left(\left(\frac{\left(\sum_{j=1}^{n_i} t_{ij}^2\right)}{2\sigma_i^2}+\frac{1}{2\varpi^2}\right)^{-1}\left(\frac{\sum_{j=1}^{n_i} t_{ij}\left(y_{ij}-\mu_i\right)}{\sigma_i^2}+\frac{\varphi}{\varpi^2}\right) \quad , \quad \left(\frac{\left(\sum_{j=1}^{n_i} t_{ij}^2\right)}{\sigma_i^2}+\frac{1}{\varpi^2}\right)^{-1}\right).$$

The prior of $\eta_i$ is Normal($\varphi$, $\varpi^2$).

2.1 The conditional posterior distribution of $\varphi|\eta_i$, $\varpi$, $f_0$, $F_0^2$ is

$$\text{Normal}\left(\left(\frac{N}{\varpi^2}+\frac{1}{F_0^2}\right)^{-1}\left(\frac{\sum_{i=1}^{N}\eta_i}{\varpi^2}+\frac{f_0}{F_0^2}\right) \quad , \quad \left(\frac{N}{\varpi^2}+\frac{1}{F_0^2}\right)^{-1}\right).$$

The prior of $\varphi$ is Normal($f_0, F_0^2$), where $f_0=0$ and $F_0^2=4$.

2.2 The conditional posterior distribution of $\varpi^2|\eta_i$, $\varphi$, $w_0$, $W_0^2$ is

$$\text{Inverse Gamma}\left(w_0+\frac{N}{2} \quad , \quad \left(\frac{1}{2}\sum_{i=1}^{N}(\eta_i-\varphi)^2+\left(W_0^2\right)^{-1}\right)^{-1}\right).$$

The prior of $\varpi^2$ is Inverse Gamma($w_0$, $W_0^2$), where $w_0=2$ and $W_0^2=0.5$.

3. The conditional posterior distribution of $\sigma_i^2|y_{ij}$, $t_{ij}$, $\mu_i$, $\eta$, $\kappa_1$, $\delta_1$ is

$$\text{Inverse Gamma}\left( \kappa_1 + \frac{n_i}{2} \quad , \quad \left( \frac{1}{2} \sum_{j=1}^{n_i} \left( y_{ij} - \left( \mu_i + t_{ij} \eta_i \right) \right)^2 + \delta_1^{-1} \right)^{-1} \right).$$

The prior of $\sigma_i^2$ is Inverse Gamma($\kappa_1$, $\delta_1$).

3.1 The conditional posterior distribution of $\kappa_1$, $\sigma_i$, $\delta_1$ is proportional to

$$\prod_{i=1}^{N} \frac{\left( \frac{1}{\delta_1 \sigma_i^2} \right)^{\kappa_1} e^{\frac{-1}{\delta_1 \sigma_i^2}}}{\Gamma(\kappa_1)},$$

which is the product of N continuous Poisson distributions with parameters of $1/(\delta_1\sigma_i^2)$. The prior of $\kappa_1$ is Uniform(10).

3.2 The conditional posterior distribution of $\delta_1|\sigma_i$, $\kappa_1$, $d_0$, $D_0$ is

$$\text{Inverse Gamma}\left( d_0 + N\kappa_1 \quad , \quad \left( \sum_{i=1}^{N} \sigma_i^{-2} + D_0^{-1} \right)^{-1} \right).$$

The prior of $\delta_1$ is Inverse Gamma($d_0$, $D_0$), where $d_0$=2 and $D_0$=2.

4. The conditional posterior distribution of $\upsilon_i$ | $t_{ij}$, $\zeta_i$, $\gamma$, $\xi$ is

$$\text{Normal}\left[\left(\frac{n_i}{2\zeta_i^2}+\frac{1}{2\xi^2}\right)^{-1}\left(\frac{\sum_{j=1}^{n_i} t_{ij}}{\zeta_i^2}+\frac{(x_i^{'}\gamma)}{\xi^2}\right) \quad , \quad \left(\frac{n_i}{\zeta_i^2}+\frac{1}{\xi^2}\right)^{-1}\right].$$

The prior of $t_{ij}$ is Normal( $x_i^{'}\gamma$ , $\xi_i^2$ ).

4.1 The conditional posterior distribution of $\gamma$ | $\upsilon_i$, $x_i$, $\xi$, $g_0$, $V_{g0}$ is

$$\text{Normal}\left(\left(V_{g_0}^{-1}+\xi^{-2}\sum x_i x_i^{'}\right)^{-1}\left(V_{g_0}^{-1}g_0+\xi^{-2}\sum \upsilon_i x_i^{'}\right),\left(V_{g_0}^{-1}+\xi^{-2}\sum x_i x_i^{'}\right)^{-1}\right).$$

The prior of $\gamma$ is Multivariate Normal($g_0$, $V_{g0}$), where $g_0$=4 and

$$V_{g_0}=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

4.2 The conditional posterior distribution of $\xi^2$| $\upsilon_i$, $x_i$, $\gamma$, $s_0$, $S_0$ is

$$\text{Inverse Gamma}\left(s_0+\frac{N}{2} \quad , \quad \left(\frac{1}{2}\sum_{i=1}^{N}\left(\upsilon_i-x_i^{'}\gamma\right)^2+S_0^{-1}\right)^{-1}\right).$$

The prior of $\xi^2$ is Inverse Gamma($s_0$, $S_0$), where $s_0$=2 and $S_0$=0.25.

5. The conditional posterior distribution of $\zeta_i^2 | t_{ij}, \upsilon_i, \kappa_2, \delta_2$ is

$$\text{Inverse Gamma}\left( \kappa_2 + \frac{n_i}{2} \quad , \quad \left( \frac{1}{2} \sum_{j=1}^{n_i} \left( t_{ij} - \upsilon_i \right)^2 + \delta_2^{-1} \right)^{-1} \right).$$

The prior of $\zeta_i^2$ is Inverse Gamma($\kappa_2$, $\delta_2$).

5.1 The conditional posterior distribution of $\kappa_2 | \zeta_i, \delta_2$ is proportional to

$$\prod_{i=1}^{N} \frac{\left( \frac{1}{\delta_2 \varsigma_i^2} \right)^{\kappa_2} e^{\frac{-1}{\delta_2 \varsigma_i^2}}}{\Gamma(\kappa_2)},$$

which is the product of N continuous Poisson distributions with parameters of

$$\left( \frac{1}{\delta_2 \varsigma_i^2} \right).$$

The prior of $\kappa_2$ is Uniform( $M_{\kappa_0}$ ), where $M_{\kappa_0}$ =10.

5.2 The conditional posterior distribution of $\delta_2 | \zeta_i, \kappa_2, d_0, D_0$ is

$$\text{Inverse Gamma}\left( d_0 + N\kappa_2 \quad , \quad \left( \sum_{i=1}^{N} \varsigma_i^{-2} + D_0^{-1} \right)^{-1} \right).$$

The prior of $\delta_2$ is Inverse Gamma($d_0$, $D_0$), where $d_0$=2 and $D_0$=2.