

The Role of Spatial Demand on Outlet Location and Pricing

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Web Appendix

WEB APPENDIX A: CONTRACTION MAPPING

We establish the existence and uniqueness of the equilibrium prices for the simulation given that $\gamma_i > 0$ and $c_j > 0$ for all i and j . Let $q \geq 0$ be a real number sufficiently large and $\bar{\gamma} = E(\gamma_i)$. We use the *constructive* approach to show that the following nonlinear vector functions

$$(A-10) \quad f_j(P) = P_j + \ln(c_j + q) - \ln\left(P_j + \left[\Omega^{-1} W(P)\right]_j + q\right)$$

$$(A-11) \quad g_j(P) = P_j - \frac{1}{\bar{\gamma}} \ln \frac{K_j}{I} + \frac{1}{\bar{\gamma}} \ln W_j(P)$$

with J_1 functions of Equation A-10 and J_2 functions of Equation A-11 ($J_1 + J_2 = J$), form a contraction mapping within a compact and convex subset of the Euclidian space R^J . Here we suppress the index t because the same rule applies to all the periods in the paper. Let $U = [0, \bar{P}]^J$ be this subset with a sufficiently large \bar{P} . Then by the contraction mapping theorem, there exist a unique solution of prices within this closed subset. Note the equivalence between our supply side model in Equation 14 in the paper and Equations A-10 and A-11 when $f_j(P) = P$.

We will write $W_j = W_j(P)$ without ambiguity and substitute the notation $\int f(\gamma_i, W_{ij}) dF(\beta_{1i}, \beta_{2i}, \gamma_{3i})$ for any function $f(\gamma_i, W_{ij})$ with $E(f(\gamma_i, W_{ij}))$.

Lemma 1: *Functions $g_j(P)$, $j=1, \dots, J$ in Equation A-11 define a contraction mapping: $U \rightarrow U$.*

Proof: Because our system is bounded, we only need establish that $\partial g_j(P) / \partial P_k > 0$

and $\sum_{k=1}^J \partial g_j(\mathbf{P}) / \partial P_k < 1$. Indeed,

$$(A-12) \quad \frac{\partial g_j(\mathbf{P})}{\partial P_k} = \frac{E[\gamma_i W_{ij} W_{ik}]}{E(\gamma_i)E(W_{ij})} > 0, \text{ for } k \neq j$$

because γ_i , W_{ij} and W_{ik} are all greater than 0. Because γ_i and W_{ij} are negatively correlated (the higher the γ_i , the greater the market share W_{i0} for the outside good), we have

$$(A-13) \quad \text{cov}(\gamma_i, W_{ij}) \leq 0 \Rightarrow E(\gamma_i)E(W_{ij}) \geq E[\gamma_i W_{ij}] \text{ and } \frac{E[\gamma_i W_{ij}]}{E(\gamma_i)E(W_{ij})} \leq 1.$$

Therefore, if $k = j$, we have

$$(A-14) \quad \frac{\partial g_j(\mathbf{P})}{\partial P_j} = \frac{E(\gamma_i)E(W_{ij}) - E[\gamma_i W_{ij}(1 - W_{ij})]}{E(\gamma_i)E(W_{ij})} \geq \frac{E[\gamma_i W_{ij}^2]}{E(\gamma_i)E(W_{ij})} > 0$$

$$\text{and } \sum_{k=1}^J \frac{\partial g_j(\mathbf{P})}{\partial P_k} = 1 - \frac{E[\gamma_i W_{ij} W_{i0}]}{E(\gamma_i)E(W_{ij})} < 1,$$

since $W_{i0} < 1$. This completes the proof.

Lemma 2: *If all the parameters in W_{ij} are finite and $c_j \geq 0$ for all j , there exist a real number $q \geq 0$ such that functions $f_j(\mathbf{P})$ in Equation A-10 for all $j=1, \dots, J$ define a contraction mapping: $U \rightarrow U$.*

Proof: We shall first show this is true for the simple case where there are only homogeneous parameters in W_{ij} and single ownership of outlets. In this case,

$[\Omega^{-1} \mathbf{W}(\mathbf{P})]_j = -\gamma^{-1}(1 - W_j)^{-1}$. We need select a number $q \geq 0$ sufficiently large such that

$$\frac{\partial f_j(\mathbf{P})}{\partial P_j} = 1 - \frac{1/(1 - W_j)}{P_j - \frac{1}{\gamma} \frac{1}{1 - W_j} + q} > 0, \text{ if } k = j$$

$$\frac{\partial f_j(\mathbf{P})}{\partial P_k} = \frac{W_j W_k / (1 - W_j)^2}{P_{ij} - \frac{1}{\gamma} \frac{1}{1 - W_j} + q} > 0, \text{ if } k \neq j,$$

$$\sum_{k=1}^J \frac{\partial f_j(\mathbf{p}_t)}{\partial P_{tk}} = 1 - \frac{1 - W_0 W_j}{P_{ij} - \frac{1}{\gamma} \frac{1}{1 - W_j} + q} < 1.$$

Note that none of W_j , $j=1,\dots,J$ can approach 0 or 1 infinitesimally because P_j can at most be \bar{P} and W_0 is greater than 0 for the outside good even when all $P_j=0$. Because all three numerators in the expressions above are bounded, we can select a real number q sufficiently large such that all the conditions are satisfied.

For the more complicated case when the parameters in W_{ij} are modeled heterogeneously and firms own multiple outlets, we can apply the same method to find a q large enough because all W_{ij} and $1-W_{ij}$ are bounded. This completes the proof.

Selecting a right q can be a challenge in real practice. However, for the Roanoke apartment data in this paper, we find that $q=0$ works perfectly because $P_j + \left[\Omega^{-1} W(P) \right]_j$ is in the neighborhood of $c_j \approx 240$, which is itself large enough to satisfy the conditions for the contraction mapping.

Lemma 3: *Let $x \in \mathbb{R}^M$ and $y \in \mathbb{R}^N$. Let $f(x, y)$ and $g(x, y)$ be two contraction mapping: $\mathbb{R}^{M+N} \rightarrow \mathbb{R}^{M+N}$. Let $f(x, y) = (f^{(1)}(x, y), f^{(2)}(x, y))$ and $g(x, y) = (g^{(1)}(x, y), g^{(2)}(x, y))$ where $f^{(1)}(x, y)$ and $g^{(1)}(x, y)$ are mappings: $\mathbb{R}^M \rightarrow \mathbb{R}^M$ and $f^{(2)}(x, y)$ and $g^{(2)}(x, y)$ are mappings: $\mathbb{R}^N \rightarrow \mathbb{R}^N$. Let $h(x, y) = (f^{(1)}(x, y), g^{(2)}(x, y))$, then $h(x, y)$ is a contraction mapping: $\mathbb{R}^{M+N} \rightarrow \mathbb{R}^{M+N}$.*

Proof: Let x_j be the j th component of vector x . Using the sup-norm metric

$$d\{x', x\} = \sup\{|x'_1 - x_1|, \dots, |x'_M - x_M|\}$$

and by the definition of contraction mappings, we have

$d\{f(x', y'), f(x, y)\} \leq K_1 d\{(x', y'), (x, y)\}$ and $d\{g(x', y'), g(x, y)\} \leq K_2 d\{(x', y'), (x, y)\}$ where $0 \leq K_1, K_2 < 1$.

$$\begin{aligned}
d\{h(x', y'), h(x, y)\} &= \sup \left\{ \left| f_1^{(1)}(x', y') - f_1^{(1)}(x, y) \right|, \dots, \left| f_M^{(1)}(x', y') - f_M^{(1)}(x, y) \right|, \right. \\
&\quad \left. \left| g_1^{(2)}(x', y') - g_1^{(2)}(x, y) \right|, \dots, \left| g_N^{(2)}(x', y') - g_N^{(2)}(x, y) \right| \right\} \\
&\leq \sup \left\{ \sup \left\{ \left| f_1^{(1)}(x', y') - f_1^{(1)}(x, y) \right|, \dots, \left| f_M^{(1)}(x', y') - f_M^{(1)}(x, y) \right|, \right. \right. \\
&\quad \left. \left. \left| f_1^{(2)}(x', y') - f_1^{(2)}(x, y) \right|, \dots, \left| f_N^{(2)}(x', y') - f_N^{(2)}(x, y) \right| \right\}, \right. \\
&\quad \left. \sup \left\{ \left| g_1^{(1)}(x', y') - g_1^{(1)}(x, y) \right|, \dots, \left| g_M^{(1)}(x', y') - g_M^{(1)}(x, y) \right|, \right. \right. \\
&\quad \left. \left. \left| g_1^{(2)}(x', y') - g_1^{(2)}(x, y) \right|, \dots, \left| g_N^{(2)}(x', y') - g_N^{(2)}(x, y) \right| \right\} \right\} \\
&= \sup \{d\{f(x', y'), f(x, y)\}, d\{g(x', y'), g(x, y)\}\} \\
&\leq \sup\{K_1, K_2\} d\{(x', y'), (x, y)\}
\end{aligned}$$

and $0 \leq \sup\{K_1, K_2\} < 1$. Therefore $h(x, y)$ is a contraction mapping.

Proposition: *The price simulation that converges eventually to a mapping: $U \rightarrow U$ with J_1 functions of Equation A-10 and J_2 functions of Equation A-11 has a unique solution.*

Proof: Lemma 1 and 2 show that $J = J_1 + J_2$ functions of either Equation A-10 or Equation A-11 are contraction mappings. Lemma 3 shows that the mixed system on $U \rightarrow U$ with J_1 functions of Equation A-10 and J_2 functions of Equation A-11 is also a contraction mapping. Because $U = [0, \bar{P}]^J$ is compact and convex, there is a unique fixed point $P \in U$ such that $P = (f(P), g(P))$.

WEB APPENDIX B: SIMULATION DESIGN

To approximate the size and structure of the data we randomly distribute 40 apartment locations in a rectangular area shown in Figure 7. For purposes of parsimony, the single complex-specific attribute considered is the presence of a garage. The single simulated location attribute is the distance to the town center, marked by a diamond in Figure 7. The utility function is Equation 1, where X_{ij} is a 0-1 indicator variable that represents the presence of the garage and X_{ts_j} is the distance to the town center. X_{ij} and X_{ts_j} do not vary over time. The diagonal distance across this rectangular region is 30 units. We set $\beta_1 = 1.0$, $\beta_2 = -0.3$, and $\gamma = 0.01$. The spatial random effect θ_{ts_j} is distributed $N(0, \sigma_0^2 R_j(\phi))$. Using the exponential covariance function for the Gaussian process, we set $\sigma_0 = 0.4$ and $\phi_0 = 0.4$. We simulate data for 10 independent periods with 10 independently sampled θ_t 's.

Figure 7

CHAIN LOCATIONS (APARTMENTS FROM THE SAME CHAIN ARE INDICATED BY THE SAME SYMBOL).

[Insert Figure 7 Here]

The prices for the 40 apartments and 10 periods are determined by the full model. The cost for each apartment is simulated using Equation 7, where Y_{ij} includes an intercept and a scalar ($Y_{ij} = (1, Y_{ij,1})$ and $\beta_3 = (\beta_{30}, \beta_{31})$) representing the size of the apartment complex's rooms (we presume larger rooms cost more to maintain). We assume three values: (1,2,3) for $Y_{ij,1}$. We set $\beta_{30} = 150$ and $\beta_{31} = 30$. ζ_{ij} is distributed $N(0, \sigma_\zeta^2 R_j(\phi))$, where $\sigma_\zeta = 10$, $\phi = 0.4$. The capacity constraint K_j for all the apartment is set to 200.

The price P_{ij} and demand Q_{ij} are generated by our demand-supply model. An iterative procedure

$$(A-15) \quad P_t^{(g+1)}(\gamma) = P_t^{(g)}(\gamma) + \ln c_t - \ln \left[P_t^{(g)} + \left[\Omega^{-1} W \left(P_t^{(g)} \right) \right]_j \right]$$

generates all the P_{ij} 's in each period. We then impute demand, $Q_{ij}(P_t)$. If Q_{ij} exceeds K_j , the price for the apartment, Equation A-15, becomes

$$(A-16) \quad P_t^{(g+1)}(\gamma) = P_t^{(g)}(\gamma) - \frac{1}{\gamma} \ln \frac{K_j}{I} + \frac{1}{\gamma} \ln W_t(P_t^{(g)})$$

All the prices P_{ij} 's are generated again and Q_{ij} 's are computed. If any Q_{ij} exceeds K_j , we replace the mapping in Equation A-15 by Equation A-16, and iterate demand and prices again. We continue in this fashion until all the equilibrium prices and quantities satisfy the capacity constraints. Using these contraction mappings, prices in our simulation converge to a equilibrium that is unique over a wide range of starting values, as proved in Web Appendix A.